

# Supersymmetric Model of a 2D Long-Range Bose Liquid

M. V. Feigel'man and M. A. Skvortsov

*L. D. Landau Institute for Theoretical Physics, Moscow 117940, RUSSIA*

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The model Hamiltonian of a two-dimensional Bose liquid (proposed earlier by Kane, Kivelson, Lee and Zhang [8] as the Hamiltonian which has Jastrow-type wavefunctions  $\Psi_0\{\mathbf{r}_i\} \propto \prod_{j>k} |\mathbf{r}_j - \mathbf{r}_k|^{2\alpha}$  as the ground-state solution), is shown to possess nonrelativistic supersymmetry. For the special value of the coupling constant  $\alpha = 1/2$  the quantum mechanics described by this Hamiltonian is shown to be equivalent to the dynamics of (complex) eigenvalues of random Gaussian ensemble of normal complex matrices. For general  $\alpha$ , an exact relation between the equal-time current-current and density-density correlation functions is obtained, and used to derive an asymptotically exact (at low wavevectors  $q$ ) spectrum of single-particle excitations beyond the superfluid ground-state (realized at low  $\alpha$ 's). The ground-state  $\Psi_0$  at very large  $\alpha$  is shown to be of "Quantum Hexatic" type, possessing long-range orientational order and quasi-long-range translational order *but with zero shear modulus*. Possible scenarios of the ground-state phase transitions as function of  $\alpha$  are discussed.

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## I. INTRODUCTION

It is commonly believed that the system of Bose particles at zero temperature may exist only in one of two possible ground-states: superfluid (SF) or crystalline (CR). Most probably it is indeed the case if the interaction between bosons is short-range (compared to the mean interparticle distance  $n^{-1/2}$ ). However the situation is much less clear if long-range and/or long-time interactions are involved. Two physical situations where the problem of a two-dimensional (2D) Bose system interacting with a 2D Maxwell gauge field were considered in [1]: classical thermodynamics of Vortex Line Liquid in High-Temperature Superconductors (see [2] for an extensive review) and low-temperature quantum properties of liquid of Holons (arising in the RVB-type theories [3] of doped Mott dielectric [4]). The arguments were given [1] in favor of existence of a new unusual ground-state which is still *liquid*, but is *not superfluid*. More recently the same problem was attacked by another approach [5], also in relation with RVB-type theories. Still the theoretical arguments in favor of the existence of such unusual ground-state are not quite conclusive; the results of different computer simulations are in mutual disagreement [6,7]. Whereas in [6] two phase transitions and an intermediate state were observed, in [7] a direct first-order transition from SF to CR state was found; note however, that the model Hamiltonians simulated in these two numerical studies were different in that the retarded interactions were presented only in [6].

There are two different questions related with this problem: i) can one invent such a ground-state in a self-consistent way? and ii) is it realized in some model with a physically realistic long-range interaction? In the present paper we try to approach the answer to the first of the above questions, using a special model of a 2D Bose liquid, introduced originally by Kane, Kivelson, Lee and Zhang (KKLZ) [8]. The KKLZ model Hamiltonian is constructed in such a way that its exact ground-state wavefunction is of the Jastrow type, i.e. it is a product of 1-particle and 2-particle dependent factors. With the special choice of the Jastrow function the main (in the long-wavelength limit) term of the KKLZ Hamiltonian is given by a pair-wise logarithmic interaction, i.e. it resembles a 2D static Coulomb potential of the real problem discussed in [1]. On the other hand, special symmetry (in fact, a kind of a supersymmetry) of the KKLZ Hamiltonian makes it possible to obtain a number of exact relations between the equal-time  $T = 0$  correlation functions, which help to analyze the general structure of the theory.

The paper is organized as follows: in Section II we derive the KKLZ Hamiltonian from the condition that it is given by the zero-Fermion sector of a supersymmetric nonrelativistic parity-conserving theory, which resembles (in the long-wavelength limit) the system of 2D bosons with logarithmic interaction. The coupling constant  $\alpha$  of our theory is proportional to the charge of 2D bosons, thus the ground-state possesses superfluidity at low  $\alpha$ . In Sec. III we analyze a special case ( $\alpha = 1/2$ ) of the obtained model and show that it is equivalent to the quantum mechanics of a Gaussian Matrix Model with normal complex matrices (in the same sense as quantum mechanics of Gaussian Hermitean Matrix Model is equivalent to 1D free fermions [9]); in Sec. IV we are back to the general case and derive an exact real-space relation between the equal-time current-current and density-density correlation functions (extending the similar results found in [10] for the Calogero-Sutherland model to a 2D case). These relations are then used in

Sec. V in order to derive a relation between momentum-space transverse current-current correlation function and the static structure factor, and in Sec. VI to find an exact form of the low-momentum excitation spectrum  $\omega(q)$  (which appears to be given by the one-loop correction to the MFA). Sec. VII is devoted to the discussion of the “crystalline” ground-state of the KKLZ Hamiltonian; we show that such a state has in fact zero shear modulus and thus should be considered as a kind of a “quantum hexatic” (QHX), in some analogy with the classical hexatic known to exist as an intermediate phase within the Halperin-Nelson picture of thermal 2D melting [11]. In Section VIII we speculate on possible properties of the additional ground-state which may be sandwiched between the SF and QHX states. Our conclusions are presented in Sec. IX. Finally, Appendix is devoted to the description a model of classical stochastic dynamics which is equivalent to our quantum-mechanical problem; we show how to use this equivalence for efficient numerical simulations of some quantum-mechanical expectation values.

## II. THE KKLZ HAMILTONIAN: SUSY STRUCTURE AND GENERAL FRAMEWORK

We will see now that the KKLZ Hamiltonian can be considered as an example of many-body generalization of the supersymmetric quantum mechanics introduced in [12,13]. We define a pair of the conjugated “supercharge” operators  $Q$  and  $Q^+$ :

$$Q = \sum_{j,\alpha} q_{j,\alpha} a_{j,\alpha}^+; \quad Q^+ = \sum_{j,\alpha} q_{j,\alpha}^+ a_{j,\alpha}; \quad (1)$$

where Fermionic operators  $a_{j,\alpha}$  and  $a_{j,\alpha}^+$  corresponds to each component of the  $j$ -th particle’s coordinate vector  $r_{j,\alpha}$  (with  $\alpha = 1, 2$ ), and

$$\begin{aligned} q_{j,\alpha} &= -i \frac{\partial}{\partial r_{j,\alpha}} + i \cdot 2\alpha \left( \sum_{k \neq j} \frac{\partial \ln |\mathbf{r}_j - \mathbf{r}_k|}{\partial r_{j,\alpha}} - \pi n r_{j,\alpha} \right), \\ q_{j,\alpha}^+ &= -i \frac{\partial}{\partial r_{j,\alpha}} - i \cdot 2\alpha \left( \sum_{k \neq j} \frac{\partial \ln |\mathbf{r}_j - \mathbf{r}_k|}{\partial r_{j,\alpha}} - \pi n r_{j,\alpha} \right). \end{aligned} \quad (2)$$

Note that the introduction of two Fermionic operators  $a_{j1}, a_{j2}$  is necessary in order to construct a P-invariant theory; if the condition of P-invariance were relaxed, the use of just one (per each particle) Fermionic operator  $c_j$  corresponding to the holomorphic coordinate  $z_j = x_j + iy_j$  would be possible. Now we define the Hamiltonian

$$\mathcal{H} = \frac{1}{2m} \{Q, Q^+\} \quad (3)$$

and note that this Hamiltonian, when acting on the state of “Fermi-vacuum” (defined by  $a_{j,\alpha}|0_F\rangle = 0$ ),  $H = \mathcal{H}|0_F\rangle$ , is equivalent to the Kane-Kivelson-Lee-Zhang (KKLZ) Hamiltonian [8] (see also [1])

$$H - E^0 = H_{\text{KKLZ}} = \frac{1}{2m} \sum_i \left\{ -\nabla_i^2 + 2\alpha \sum_{j \neq i} \nabla_i^2 \ln |\mathbf{r}_i - \mathbf{r}_j| + 4\alpha^2 \sum_{j \neq i, k \neq i} \nabla_i \ln |\mathbf{r}_i - \mathbf{r}_j| \nabla_i \ln |\mathbf{r}_i - \mathbf{r}_k| + (2\pi\alpha n)^2 \mathbf{r}_i^2 \right\}, \quad (4)$$

where  $E^0 = -2\pi\alpha n N[1 + \alpha(N-1)]/m$ . The corresponding ground-state wavefunction is given by

$$\Psi_0(\mathbf{r}_1, \dots, \mathbf{r}_N) = \text{const} \cdot \prod_{j>k} |\mathbf{r}_j - \mathbf{r}_k|^{2\alpha} e^{-\pi\alpha n \sum_i r_i^2}. \quad (5)$$

Alternatively, the Hamiltonian can be written directly in terms of the  $q_{j,\alpha}$  operators:

$$H = \frac{1}{2m} \sum_j q_{j,\alpha}^+ q_{j,\alpha}. \quad (6)$$

Each of the  $q_{j,\alpha}$  operators annihilates the ground state (5), so the ground-state energy corresponding to  $H$  is zero. The Hamiltonian (4) contains 2-body and 3-body interactions; in terms of the local particle’s density  $\rho(\mathbf{r})$  the corresponding potential energy is given by

$$U = \frac{1}{2} \int d^2r_1 d^2r_2 \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) V_2(\mathbf{r}_1 - \mathbf{r}_2) + \int d^2r_1 d^2r_2 d^2r_3 \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) \rho(\mathbf{r}_3) V_3(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_1 - \mathbf{r}_3), \quad (7)$$

where

$$V_2(\mathbf{r}) = \frac{4\pi\alpha}{m} \delta(\mathbf{r}) + \frac{4\alpha^2}{m} \frac{1}{r^2}; \quad V_3(\mathbf{r}, \mathbf{r}') = \frac{2\alpha^2}{m} \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2 r'^2}.$$

Writing  $\rho(\mathbf{r}_1) = n + \delta\rho(\mathbf{r}_1)$  where  $n$  is the mean density, and integrating over  $\mathbf{r}_1$  in the three-body term of Eq. (7) we find effective two-body potential

$$V_2^{\text{eff}}(r) = -\frac{g^2}{2\pi} \ln r + V_2(r), \quad (8)$$

where we introduced the Coulomb coupling constant  $g = 4\pi\alpha\sqrt{n/m}$  in the same way as in [1]. Now the potential energy can be rewritten in terms of the density variations  $\delta\rho = \rho - n$ , with the replacements  $\rho(\mathbf{r}) \rightarrow \delta\rho(\mathbf{r})$  and  $V_2(\mathbf{r}) \rightarrow V_2^{\text{eff}}(\mathbf{r})$  made in Eq. (7). On large space scales the main term in the potential energy is the one containing  $\ln r$ , which leads to the existence of a plasmon gap in the longitudinal branch of the excitation spectrum:  $\omega(q=0) = \omega_0 = \sqrt{ng^2/m}$ .

Below we will be interested in the presence of superfluidity in the ground state defined in Eq. (5). We define superfluid density  $n_s$  via the variation of the ground-state energy  $E_0$  (do not be confused by the different quantities  $E_0$  and  $E^0$  in Eq. (4)) with respect to the external (source) gauge field  $\mathbf{a}$  minimally coupled to the Hamiltonian (4) via the replacement  $\nabla_j \rightarrow \nabla_j + i\mathbf{a}(\mathbf{r}_j)$  in the kinetic energy term only:

$$\frac{n_s}{m} = \frac{\partial^2 E_0\{\mathbf{a}\}}{\partial \mathbf{a}_\alpha(\mathbf{q}) \partial \mathbf{a}_\beta(-\mathbf{q})} \Big|_{q \rightarrow 0} \cdot (\delta_{\alpha\beta} - \hat{q}_\alpha \hat{q}_\beta) = \frac{n}{m} - \mathcal{P}(q \rightarrow 0, \omega = 0), \quad (9)$$

where the polarization function  $\mathcal{P}(q, \omega = 0)$  is related to the transverse current-current correlation function  $\Pi_\perp(q, \omega)$  by the zero- $T$  FDT relation:

$$\mathcal{P}(q, \omega = 0) = \frac{2}{\pi} \int_0^\infty \frac{\Pi_\perp(q, \omega)}{\omega} d\omega. \quad (10)$$

Below in Sec. IV we will find the relation between the *instantaneous* transverse current-current correlation function  $\Pi_\perp(q) = \frac{1}{\pi} \int_0^\infty \Pi_\perp(q, \omega) d\omega$  and the density-density correlation function. Note that the replacement  $\nabla \rightarrow \nabla + i\mathbf{a}$  made in all terms in Eq. (4) would retain SUSY structure of the theory and therefore would not change the ground-state energy  $E_0 = 0$ .

The ground-state wavefunction is given by Eq. (5) for any value of the coupling constant  $\alpha$ , however this multiparticle wavefunction describes at least two ground-states which are very different physically. Namely, at  $\alpha \leq 1$  the ground-state is a superfluid [8] and its superfluid density  $n_s$  is equal to the total particle's density  $n$ . However, superfluidity is not guaranteed by the form of the ground-state wavefunction only: at very large values of  $\alpha$  (larger than  $\alpha_{\text{cr}} \approx 35$ ) the ground state is expected to possess translational symmetry breaking realized in the formation of a triangular lattice [23] (in fact only quasi-long Berezinsky-type [14] translational order is formed). Such a state does not possess superfluidity,  $n_s = 0$ . Therefore at least one quantum phase transition, between the states with nonzero and zero  $n_s$ , should happen as the parameter  $\alpha$  increases. There is also a possible room for the second phase transition. To see it, note that the Galilei invariance of the Hamiltonian (4) implies (at  $T = 0$ ) the absence of a normal density:  $n_n = n - n_s = 0$ . It means that nonzero  $n_n$  can appear due to spontaneous breakdown of the Galilei invariance, which may happen simultaneously with the lattice formation (so that  $n_s$  jumps from  $n$  to zero), or at some other critical value  $\alpha_G < \alpha_{\text{cr}}$ . Moreover, there are no special reason for the superfluid density to go to zero at  $\alpha = \alpha_{\text{cr}}$ , as it may happen at  $\alpha \in [\alpha_G, \alpha_{\text{cr}}]$ . Finally, there is one more symmetry, the orientational one, which can be spontaneously broken by the ground-state at  $\alpha = \alpha_O \in [\alpha_G, \alpha_{\text{cr}}]$ . Below we start our consideration from the small- $\alpha$  range where no symmetry is broken, then proceed (in Sec. VII) to the region of largest  $\alpha > \alpha_{\text{cr}}$ , and at the end we present some speculations on the system's behavior in the intermediate range of  $\alpha$ .

### III. THE SPECIAL CASE $\alpha = 1/2$ : MAPPING TO QUANTUM MECHANICS OF NORMAL MATRICES

We will show now that the dynamics of a Gaussian ensemble of  $N \times N$  complex matrices ( $N \rightarrow \infty$ ) constrained by the condition  $[M, M^+] = 0$  (which is the definition of a normal matrix) is intimately related to the many-body

quantum mechanics defined by the Hamiltonian (4) with  $\alpha = 1/2$ . This relation is of the same origin as the well-known now relation [9] between dynamics of the Dyson matrix ensembles and the Calogero-Sutherland model of many-body quantum mechanics in 1 dimension, where the coordinates of 1D particles are just eigenvalues of the corresponding matrices. Eigenvalues of a normal matrix are, in general, complex numbers, which can be considered as coordinates of particles living in a 2D space. Thus our goal here is to show how it is possible to perform functional integral over most part of variables characterizing dynamics of a normal matrix and reduce the problem to the quantum mechanics for the set of its complex eigenvalues  $z_i, \bar{z}_i$ . Our derivation follows closely the one presented in [9] (see also [15]) for three Dyson ensembles. Consider the partition function

$$Z = \int \mathcal{D}\hat{M}(\tau) \mathcal{D}\hat{M}^+(\tau) \exp[-S(\hat{M}, \hat{M}^+)] \prod_{\tau} \delta([\hat{M}(\tau), \hat{M}^+(\tau)]) \quad (11)$$

with the flat measure of integration and the Gaussian action

$$S(\hat{M}, \hat{M}^+) = \int \left[ \frac{1}{4} \text{Tr} \left( \frac{\partial \hat{M}^+}{\partial \tau} \frac{\partial \hat{M}}{\partial \tau} \right) + \text{Tr} (\hat{M}^+ \hat{M}) \right] d\tau. \quad (12)$$

Note now that any normal matrix can be diagonalized by an appropriate unitary transformation:  $\hat{M} = \hat{U}^+ \hat{Z} \hat{U}$ , where  $\hat{Z}$  is diagonal with the elements  $z_i$ . Therefore we can change variables in the functional integral (11):

$$\mathcal{D}\hat{M}(\tau) \mathcal{D}\hat{M}^+(\tau) = J\{z_i\} \prod_i dz_i d\bar{z}_i \mathcal{D}\hat{U}, \quad (13)$$

where the Jacobian of the transformation is given [16] by the square of the absolute value of the Vandermonde determinant:  $J\{z_i\} = \text{const} |\Delta\{z_i\}|^2 = \text{const} \prod_{i>j} |z_i - z_j|^2$ . Consider now the Hamiltonian corresponding to the above functional integral in the transformed variables:

$$H = -\frac{2}{\sqrt{g}} \sum_i \left[ \frac{\partial}{\partial z_i} \sqrt{g} \frac{\partial}{\partial \bar{z}_i} + \frac{\partial}{\partial \bar{z}_i} \sqrt{g} \frac{\partial}{\partial z_i} \right] + \sum_i |z_i|^2 + \tilde{H}. \quad (14)$$

Here the first sum over  $i$  constitutes a Laplacian operator in the curved space with the square root of the determinant of the metric tensor  $\sqrt{g} = |\Delta\{z_i\}|^2$ , the second one comes from the “potential energy”  $\text{Tr} (\hat{M}^+ \hat{M})$ , and the last term corresponds to rotations of angular (i.e. related to unitary transformations) variables, and thus vanishes when acting on any singlet ( $\hat{U}$ -invariant) state. The states with nonzero angular momenta have much higher energy [17] and thus are irrelevant, therefore we will omit  $\tilde{H}$  below. Let us note now that the rest of the Hamiltonian (14) can be rewritten in the form

$$-\frac{4}{|\Delta\{z_i\}|} \sum_i \frac{\partial^2}{\partial z_i \partial \bar{z}_i} |\Delta\{z_i\}| + V\{z_i, \bar{z}_i\}, \quad (15)$$

where the potential energy  $V$  is given by

$$V\{z_i, \bar{z}_i\} = \sum_i \sum_{j \neq i, k \neq i} \frac{1}{(z_i - z_k)(\bar{z}_i - \bar{z}_j)} + 4 \sum_{i \neq j} \frac{\partial^2}{\partial z_i \partial \bar{z}_i} \log |z_i - z_j| + \sum_i |z_i|^2. \quad (16)$$

After redefinition of the wavefunction  $\Psi \rightarrow |\Delta\{z_i\}| \Psi$  and rewriting  $z_i$  and  $\bar{z}_i$  via real 2-vectors  $\mathbf{r}_i$ , we are back to the Hamiltonian defined in Eq. (4) (apart from the constant term  $E^0$ ) with  $\alpha = 1/2$ ,  $m = 1/2$  and  $n = 1/\pi$ . Evidently, the values of  $m$  and  $n$  give just the units of time and length in the problem, and can be rescaled arbitrary. The equilibrium probability distribution for the eigenvalues (for the density of eigenvalues equal to  $n$ ) is:

$$P\{z_i\} = |\Psi_0\{z_i\}|^2 = \text{const} \cdot \prod_{j>k} |z_j - z_k|^2 e^{-\pi n \sum_i r_i^2} \quad (17)$$

and coincides (up to a constant normalizing factor) with the corresponding distribution for the general complex ensemble [18,16] (cf. also recent preprints [19,20]). Unfortunately it does not seem possible to establish an exact relation between the dynamics of general complex matrices and some Hamiltonian problem defined in terms of eigenvalues only; the problem is that a general complex matrix cannot be diagonalized by a unitary transformation and thus

angular variables cannot be eliminated from the dynamical problem [21]. It is only the stationary distribution of its eigenvalues  $P_{\text{complex}}\{z_i\}$  which coincides (up to a constant factor) with such a distribution for a normal matrix ensemble given by Eq. (17). However, the relations obtained above indicate that the correspondence between the KKLZ Hamiltonian and normal matrices is much deeper than just on the level of the stationary distribution. The supersymmetric structure of this Hamiltonian and the obtained relation with a relatively simple matrix model lead us to believe that the model under discussion might be a rather direct generalization of the Calogero-Sutherland model for a 2D case, and thus might even be amenable for the exact solution. However presently we are not aware of any exact way to treat normal matrix dynamics, and therefore will turn in the next Sections to some more elementary methods, which still make it possible to find some exact relations between the equal-time correlation functions of the KKLZ model at arbitrary value of the parameter  $\alpha$ .

#### IV. THE RELATION BETWEEN THE CURRENT-CURRENT AND DENSITY-DENSITY CORRELATION FUNCTIONS

The SUSY structure of the Hamiltonian (4) makes it possible to find, using the method used previously by Taniguchi, Shastry and Altshuler for the Calogero-Sutherland model [10] (see also recent preprint [22]), an exact relation between the regular parts of the equal-time current-current  $\Pi_{\alpha\beta}^{\text{reg}}(\mathbf{r})$  and density-density  $K(r)$  correlation functions defined as

$$\Pi_{\alpha\beta}^{\text{reg}}(\mathbf{r} - \mathbf{r}') = \langle J_\alpha(\mathbf{r}) J_\beta(\mathbf{r}') \rangle^{\text{reg}}; \quad K(\mathbf{r} - \mathbf{r}') = \langle \rho(\mathbf{r}) \rho(\mathbf{r}') \rangle^{\text{reg}}. \quad (18)$$

In the definitions (18) the singular (i.e. proportional to  $\delta$ -functions) terms in the current and density correlators are excluded (i.e.  $\mathbf{r} \neq \mathbf{r}'$ ). It means, in particular, that the correlator  $K(\mathbf{r})$  is related to the inverse Fourier-transform  $s(\mathbf{r})$  of the static structure factor  $S(\mathbf{p}) = \int (d\omega) S(\omega, \mathbf{p})$  as follows:

$$K(\mathbf{r} - \mathbf{r}') = \langle 0 | \sum_{j \neq k} \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{r}' - \mathbf{r}_k) | 0 \rangle = s(\mathbf{r} - \mathbf{r}') - n \delta(\mathbf{r} - \mathbf{r}') + n^2, \quad (19)$$

where  $n$  is the mean density of particles; the last term in (19) is due to the fact that  $K(\mathbf{r})$  (contrary to  $s(\mathbf{r})$ ) is the reducible correlation function). Similarly, the relation between the full current-current correlation function and its regular part is:

$$\begin{aligned} \Pi_{\alpha\beta}^{\text{reg}}(\mathbf{r} - \mathbf{r}') &= \frac{(-i)^2}{4m^2} \langle 0 | \sum_{j \neq k} \left( \frac{\partial}{\partial r_{j,\alpha}} \delta(\mathbf{r} - \mathbf{r}_j) + \delta(\mathbf{r} - \mathbf{r}_j) \frac{\partial}{\partial r_{j,\alpha}} \right) \left( \frac{\partial}{\partial r_{k,\beta}} \delta(\mathbf{r}' - \mathbf{r}_k) + \delta(\mathbf{r}' - \mathbf{r}_k) \frac{\partial}{\partial r_{k,\beta}} \right) | 0 \rangle \\ &= \Pi_{\alpha\beta}(\mathbf{r} - \mathbf{r}') - \frac{n}{m} E_{\text{kin}} \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}'), \end{aligned} \quad (20)$$

where  $E_{\text{kin}} = \frac{m}{2N} \sum_i \langle 0 | v_i^2 | 0 \rangle$  is the mean ground-state kinetic energy per particle (in  $D$  dimensions the last term in Eq. (20) should be multiplied by  $2/D$ ).

The trick to prove the relation between  $\Pi_{\alpha\beta}^{\text{reg}}(\mathbf{r})$  and  $K(\mathbf{r})$  is to substitute, following [10], the standard definition of the current operator by the equivalent expression

$$J_\alpha(\mathbf{r}) = \frac{1}{2m} \sum_i (q_{i,\alpha}^+ \delta(\mathbf{r} - \mathbf{r}_i) + \delta(\mathbf{r} - \mathbf{r}_i) q_{i,\alpha}). \quad (21)$$

Then, using the commutation relations

$$[q_{i,\alpha}, q_{j,\beta}^+] = 4\alpha \left[ (1 - \delta_{ij}) D_{\alpha\beta}(\mathbf{r}_i - \mathbf{r}_j) - \delta_{ij} \sum_{l \neq i} (D_{\alpha\beta}(\mathbf{r}_i - \mathbf{r}_l) - \pi n \delta_{\alpha\beta}) \right] \quad (22)$$

with  $D_{\alpha\beta}(\mathbf{r}) = \nabla_\alpha \nabla_\beta \ln r$ , and identities  $q_{i,\alpha} | 0 \rangle = 0$ , we get

$$\Pi_{\alpha\beta}^{\text{reg}}(\mathbf{r}) = \frac{\alpha}{m^2} \frac{\delta_{\alpha\beta} - 2\hat{r}_\alpha \hat{r}_\beta}{r^2} K(\mathbf{r}). \quad (23)$$

The relation (23) is applicable for any value of the coupling constant  $\alpha$  (remember that  $\alpha \ll 1$  corresponds to the superfluid ground state, whereas at very large  $\alpha \geq \alpha_{\text{cr}} \approx 35$  the Wigner-type crystal is formed [23]). One may wonder whether it is allowed to use the relation (23) at  $\alpha \geq \alpha_{\text{cr}}$  since in course of its derivation we have used space homogeneity of the correlation functions, which usually is not present in the crystalline state; however we will see below that the “crystalline” state present in our problem actually does not possess a true positional long-range order and thus does not break space homogeneity.

## V. TRANSVERSE CURRENTS AND “INSTANTANEOUS” MAGNETIC RESPONSE

We are going to obtain the momentum-space representations for the longitudinal and transverse current-current correlators defined as  $\Pi_{\alpha\beta}(\mathbf{q}) = \hat{q}_\alpha \hat{q}_\beta \Pi_{\parallel}(\mathbf{q}) + (\delta_{\alpha\beta} - \hat{q}_\alpha \hat{q}_\beta) \Pi_{\perp}(\mathbf{q})$ . Performing the Fourier transformation of the expression (23) and using the relation (20), we get

$$\Pi_{\alpha\beta}(\mathbf{q}) = \frac{2\pi\alpha}{m^2} \int \frac{d^2p}{(2\pi)^2} \frac{(p+q)_\alpha (p+q)_\beta}{(\mathbf{p}+\mathbf{q})^2} \tilde{K}(\mathbf{p}) + \frac{2\pi\alpha}{m^2} n^2 \frac{q_\alpha q_\beta}{q^2} + \frac{n}{m} E_{\text{kin}} \delta_{\alpha\beta}. \quad (24)$$

Here  $\tilde{K}(\mathbf{p}) = S(\mathbf{p}) - n$  is the Fourier-transform of  $K(\mathbf{r}) - n^2$ . In order to obtain the transverse polarization function  $\Pi_{\perp}(\mathbf{q})$  one has to contract Eq. (24) with the transverse projector  $\delta_{\alpha\beta} - \hat{q}_\alpha \hat{q}_\beta$ :

$$\Pi_{\perp}(\mathbf{q}) = \frac{2\pi\alpha}{m^2} \int \frac{d^2p}{(2\pi)^2} \left[ 1 - \frac{[(\mathbf{p}+\mathbf{q})\mathbf{q}]^2}{(\mathbf{p}+\mathbf{q})^2 q^2} \right] \tilde{K}(\mathbf{p}) + \frac{n}{m} E_{\text{kin}}. \quad (25)$$

The  $q \rightarrow 0$  limit of the frequency-dependent  $\Pi_{\perp}(\mathbf{q}, \omega)$  is zero for any  $\omega$ , leading to the same result for the equal-time correlator:  $\Pi_{\perp}(\mathbf{q} \rightarrow 0) = 0$ . This condition makes it possible to determine  $E_{\text{kin}}$ :

$$E_{\text{kin}} = -\frac{\pi\alpha}{mn} \int \frac{d^2p}{(2\pi)^2} \tilde{K}(\mathbf{p}) = -\frac{\pi\alpha}{mn} [K(r=0) - n^2]. \quad (26)$$

Now,  $K(r=0) = 0$  since the ground state (5) forbids 2 particles to be at the same point:  $\Psi_0(\dots, \mathbf{r}, \dots, \mathbf{r}, \dots) = 0$ . Consequently, the expectation value of the kinetic energy in the ground state is given by

$$E_{\text{kin}} = \frac{\pi\alpha n}{m} = \frac{\omega_0}{4}, \quad (27)$$

where  $\omega_0 = \sqrt{ng^2/m}$  is the plasma frequency. The relation (27) was checked numerically (see Appendix for the details) and found to be obeyed with a good accuracy for all values of  $\alpha$ , including those where Quantum Hexatic phase is formed, where the Galilei and rotational symmetries are both broken. We will discuss the implications of this result below, in Sec. VII.

Then, Eq. (25) may be rewritten as

$$\Pi_{\perp}(\mathbf{q}) = -\frac{2\pi\alpha}{m^2} \int \frac{d^2p}{(2\pi)^2} \left[ \frac{[(\mathbf{p}+\mathbf{q})\mathbf{q}]^2}{(\mathbf{p}+\mathbf{q})^2 q^2} - \frac{1}{2} \right] (S(\mathbf{p}) - n). \quad (28)$$

If the orientational symmetry is unbroken (which is the case for not very large  $\alpha$ ), the correlation function  $S(\mathbf{p})$  depends on the absolute value of the vector  $\mathbf{p}$  only. Then the angular integral in Eq. (28) can be easily calculated, the result is:

$$\Pi_{\perp}(q) = -\frac{\alpha}{2m^2} \int_0^q (S(p) - n) \left( 1 - \frac{p^2}{q^2} \right) p dp. \quad (29)$$

At small momenta  $p$  the structure factor  $S(p)$  goes to zero:  $S(p \rightarrow 0) = np^2/2m\omega_0$ , therefore the low-momenta limit of the transverse correlator reduces to

$$\Pi_{\perp}(q \rightarrow 0) = \frac{\alpha n}{8m^2} \cdot q^2. \quad (30)$$

In the high- $q$  limit the integral in Eq. (29) reduces to the one of Eq. (26):

$$\Pi_{\perp}(q \rightarrow \infty) = \frac{\pi\alpha n^2}{m^2}. \quad (31)$$

In the special case  $\alpha = 1/2$  the current-current correlation function can be calculated explicitly. Indeed, the real-space density-density correlation function is known [18,16]:

$$K_{1/2}(r) = n^2 [1 - \exp(-\pi n r^2)] \quad (32)$$

(this expression was derived by direct integration of the total distribution function (17) over positions of all but two particles). The corresponding structure factor is

$$S_{1/2}(p) = n[1 - \exp(-p^2/4\pi n)], \quad (33)$$

which leads, via integration in Eq. (29), to

$$\Pi_{\perp,1/2} = \frac{\pi n^2}{2m^2} \left[ 1 - \frac{1 - \exp(-q^2/4\pi n)}{q^2/4\pi n} \right]. \quad (34)$$

Note that the current-current correlations in the Calogero-Sutherland model were shown to be related to the correlations of eigenfunctions of the corresponding random matrices [10]; we expect that a similar relation might exist between the correlation function (34) and some eigenfunctions correlations in the random ensemble of normal matrices.

We have employed numerical simulations (cf. Appendix) to obtain the structure factor  $S(p)$  for several different values of  $\alpha$ , and then used the obtained results to compute, using Eq. (29), the transverse current-current correlation function  $\Pi_{\perp}(p)$ . The results are shown in Fig. 1. The peaks at  $p \sim 2\pi n^{1/2}$  correspond to a roton minimum in the excitation spectrum that becomes deeper as  $\alpha$  increases.

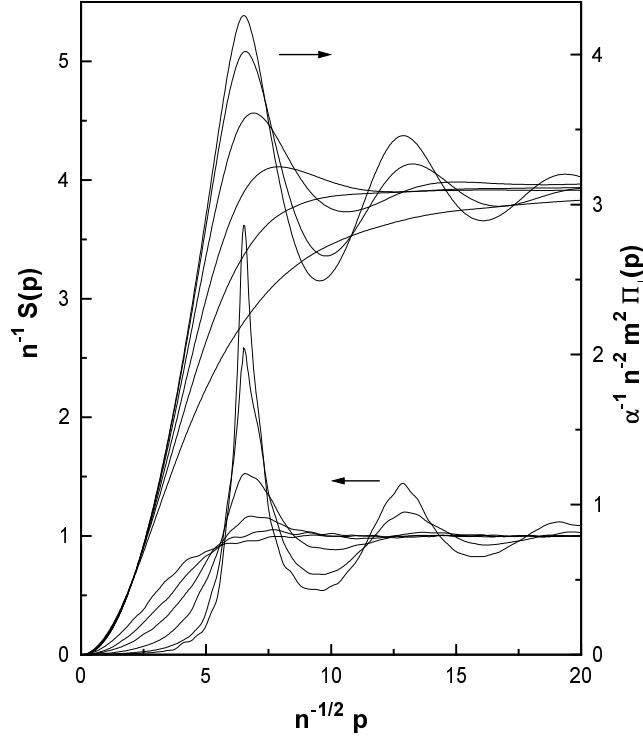


FIG. 1. The structure factor  $S(p)$  (left axis) and the transverse current-current correlator  $\Pi_{\perp}(p)$  (right axis) computed numerically by the mapping of the Schrödinger equation with the Hamiltonian (4) onto a Langevin dynamics of classical particles (see Appendix). The curves correspond to  $\alpha = 0.5, 1, 2, 5, 15$  and  $30$ . The increase of  $\alpha$  leads to the increase of the peak height at  $p \sim 2\pi n^{1/2}$ .

## VI. LONGITUDINAL CURRENTS AND LOW- $Q$ PLASMON DISPERSION.

Now we turn to the calculation of the longitudinal correlator  $\Pi_{\parallel}(\mathbf{q})$ . Contracting Eq. (24) with the longitudinal projector and using (26), we obtain

$$\Pi_{\parallel}(\mathbf{q}) = \frac{2\pi\alpha}{m^2} n^2 - \Pi_{\perp}(\mathbf{q}) = \frac{2\pi\alpha}{m^2} n^2 + \frac{\alpha}{2m^2} \int_0^q (S(p) - n) \left( 1 - \frac{p^2}{q^2} \right) p dp. \quad (35)$$

Alternatively, the value  $\Pi_{\parallel}(0)$  may be found with the help of the continuity equation that leads to the relation  $q^2 \Pi_{\parallel}(q, \omega) = \omega^2 S(q, \omega)$  for the frequency-dependent correlators. At  $q \rightarrow 0$  the whole spectral weight is saturated by the plasmon excitations with  $\omega = \omega_0$ , so we get

$$q^2 \Pi_{\parallel}(q) = \int \frac{d\omega}{2\pi} \omega^2 S(q, \omega) \xrightarrow{q \rightarrow 0} \frac{nq^2 \omega_0}{2m} \quad (36)$$

leading to  $\Pi_{\parallel}(0) = n\omega_0/2m$  in agreement with (35).

Eq. (35) can be used to find the low- $q$  dispersion of the plasmon energy  $\omega(q)$ : using the continuity equation  $q^2 \Pi_{\parallel}(q, \omega) = \omega^2 S(q, \omega)$  together with the  $f$ -sum rule  $\int (d\omega) \omega S(q, \omega) = nq^2/2m$ , we convert Eq. (35) into the form

$$\frac{\partial}{\partial q^2} \tilde{\omega}(q) = \frac{\alpha}{mn} \frac{1}{q^4} \int_0^q (S(p) - n) p^3 dp, \quad (37)$$

where

$$\tilde{\omega}(q) = \frac{\int \omega^2 S(q, \omega) d\omega}{\int \omega S(q, \omega) d\omega}. \quad (38)$$

As long as the spectrum  $\omega(q)$  is nondecaying [i.e.  $\text{Im} \omega(q) = 0$ ],  $\tilde{\omega}(q)$  coincides with  $\omega(q)$ .

Finally, using the relation  $S(q) = nq^2/2m\omega(q)$  valid at  $q \rightarrow 0$  and neglecting small  $S(q)$  compared to  $n$  in the r.h.s. of Eq. (37), we arrive at

$$\tilde{\omega}(q) - \omega_0 = -\alpha \frac{q^2}{4m}. \quad (39)$$

Eq. (39) is very similar to the one derived [24] in the lowest-order perturbation theory (over  $\alpha$ ) for the purely Coulomb 2D Bose-liquid (the result of [24] contains extra numerical factor  $\approx 0.8$  in the r.h.s. of Eq. (39)). Let us emphasize, however, that in the present model we are able to derive the relation (39) for *any* value of  $\alpha$ , the only limitation was the low- $q$  limit. In the field-theoretic language, the low-momentum excitation spectrum of the KKLZ Hamiltonian is given by the first-loop term in the diagram expansion. We believe that this remarkable property is due to the supersymmetry of the model, leading somehow to the cancellation of all higher-order diagrams. It is not yet clear to us whether the one-loop calculation is exact not only for the low- $q$  spectrum, but for the spectrum and the correlation functions in general. To study this issue one needs to develop an explicitly supersymmetric second-quantized representation and the corresponding diagram technique for the KKLZ model.

As long as  $\alpha \leq 1$ , the quadratic dispersion  $\omega(q)$  given by Eq. (39) produces a relatively weak minimum between  $\omega_0$  and the free-particle limit  $\omega_{\text{free}}(q) = q^2/2m$ . Such a spectrum is consistent with a totally superfluid ( $n_s = n$ ) ground state. With the increase of  $\alpha$  the minimum in the spectrum becomes deeper; for a qualitative description of the spectrum one can use a kind of an interpolation formula

$$\omega^2(q) \approx \omega_0^2 - \omega_0 \alpha q^2/2m + (q^2/2m)^2. \quad (40)$$

The minimum of the r.h.s. of Eq. (40) touches zero at  $\alpha = 2$ . It can be considered as an indication of possible spectrum instability and phase transition(s) in the large- $\alpha$  region. However the nature of Eq. (40) as interpolation formula of unknown (at large  $\alpha$ ) accuracy makes it impossible to use it for any exact conclusion about the behavior of the spectrum. In the next Section we will try to consider this problem starting from the opposite limit of extremely large  $\alpha$ .

## VII. “CRYSTALLINE” GROUND STATE: QUANTUM HEXATIC WITHOUT SHEAR MODULUS

At very large values of  $\alpha > \alpha_{\text{cr}} \approx 35$  the ground state is expected to possess symmetry breaking realized in the formation of triangular lattice [23]. The static structure factor  $S(p)$  can be determined knowing the ground state wavefunction only. Its calculation involves integration over all particle’s coordinates  $\mathbf{r}_i$  with the weight given by  $\Psi_0^2\{\mathbf{r}_i\}$ , which is formally equivalent to the calculation of the structure factor of a classical 2D crystal with a logarithmic repulsion between particles  $V(r) = 4\alpha \cdot \ln r$  and “temperature”  $\mathcal{T} = 1$  (cf. Appendix). Such a classical crystal has a finite shear modulus  $\mu_{\text{cl}} \sim \alpha n$  and is expected to undergo dislocation-mediated melting when  $\alpha$  decreases below some critical value  $\alpha_1$ . At  $\alpha > \alpha_1$  it is stable with respect to dislocation formation, but does not possess long-range translational order (as any other 2D crystal at finite temperature). It means that the structure factor  $S(p)$  does not have Bragg peaks near the principal inverse lattice vectors  $\mathbf{p}_a$ , but shows only power-low singularities:

$$S(p) \sim |\mathbf{p} - \mathbf{p}_a|^{\eta-2} \quad (41)$$



with  $\eta \sim \alpha^{-1}$ . Let us now go back to our original problem, which is a quantum 2D crystal. Suppose that it has finite shear modulus  $\mu$  (now without subscript ‘cl’), then the spectrum of the transverse sound would be linear, as usual:  $\omega(q) \sim \sqrt{\mu}q$  where  $q$  is the wavevector of a shear deformation on a top of the crystal state. (In terms of momenta  $\mathbf{p}$  of the original Bose particles it corresponds to  $\omega_{\perp}(\mathbf{p}) \sim |\mathbf{p} - \mathbf{p}_a|$  for any  $a$ ). Now we can calculate the zero-point fluctuations of lattice displacements  $\langle u^2 \rangle \sim \int d^2q \omega^{-1}(q)$  and find it finite, which means a finite Debye-Waller factor and thus the presence of Bragg peaks in the structure factor  $S(p)$  – in contradiction with the above-obtained Eq. (41). The only way out of this contradiction is to suppose that the quantum solid with our very special interaction given by Eq. (4) has actually *zero* shear modulus:  $\mu = 0$ . More precisely, we need to suppose that  $\omega(q) \propto q^2$ , which leads then to a logarithmically divergent  $\langle u^2 \rangle$ , in agreement with Eq. (41). It means that the “crystalline” ground state expected to exist at  $\alpha > \alpha_1$  should be more adequately called “Quantum Hexatic” (QHX), which is (in close analogy to the classical hexatic phase introduced by Halperin and Nelson within their picture of dislocation-mediated thermal melting of 2D crystals [11]) the phase without shear modulus but still with anisotropy of the excitation spectrum:

$$\omega_{\perp}(\mathbf{p}) \sim (\mathbf{p} - \mathbf{p}_a)^2 / M \quad (42)$$

( $M$  is some parameter with dimensionality of mass), whereas in the isotropic liquid excitation spectrum depends on the momentum absolute value  $|\mathbf{p}|$  only.

The absence of a shear modulus in the high- $\alpha$  ground-state of the Hamiltonian (4) can also be shown by direct calculation which can be conveniently performed representing potential energy of interaction between particles as

$$U_{\text{KKLZ}} = \frac{2\alpha^2}{m} \sum_i \left( \sum_{j \neq i} \frac{\mathbf{r}_j - \mathbf{r}_i}{(\mathbf{r}_j - \mathbf{r}_i)^2} \right)^2 + \frac{2\pi\alpha}{m} \sum_{i \neq j} \delta(\mathbf{r}_i - \mathbf{r}_j) \quad (43)$$

Now one can calculate the variation of the potential energy (actually the second sum in (43) does not contribute) corresponding to an infinitesimal uniform shear deformation  $\epsilon_{\alpha\beta}$  of the regular lattice. Then one finds that it does not contain the term proportional to  $\epsilon_{\alpha\beta}^2$ , i.e. the expansion starts from the term  $(\nabla \epsilon_{\alpha\beta})^2$ , which means that  $\mu(q) \propto q^2$ .

The absence of a genuine crystalline state in the KKLZ model at arbitrary strong interaction  $\alpha \rightarrow \infty$  stems from its very special form and certainly is not present in a generic 2D Bose system. However, in a more realistic system QHX may exist as an intermediate ground-state realized at moderately large repulsion.

Let us consider now the response of the QHX state to a magnetic field  $h$ . Since particles in this state are almost localized, we can neglect (in a first approximation) the multi-particle exchange contributions and estimate susceptibility as  $\chi \sim \frac{n}{m} \langle u^2 \rangle \sim \frac{M}{m} \ln(nL^2)$  (where  $L$  is the system’s size). Comparing this expression with the magnetic susceptibility of a superfluid, which is proportional to  $n_s L^2$ , one concludes that  $n_s = 0$  in the QHX state.

Now we need to address some apparent paradox: i) according to Eq. (10), normal density  $n_n = n - n_s$  is proportional to the integral  $\int_0^\infty \Pi_{\perp}(q \rightarrow 0, \omega) d\omega / \omega$  where the expression under the integral is positive; ii) the equal-time current-current correlation function  $\Pi_{\perp}(q) = \int_0^\infty \Pi_{\perp}(q, \omega) d\omega / \pi$  vanishes at  $q \rightarrow 0$  (cf. Eq. (27)); iii) then, how it is possible to get nonzero  $n_n$  (and even cancel  $n_s$  completely)? In fact, to get nonzero  $n_n$  we need to allow for the singular behavior of the function  $\Pi_{\perp}(q, \omega)$  at low  $q, \omega$  (in other words, the limiting procedure  $q \rightarrow 0$  and integration in Eq. (10) should not commute with each other). As a simple example one can consider  $\Pi(q, \omega) \propto f(q^2/m\omega)$  with  $f(x) = x/(1+x^2)$ . However, such a behavior of  $\Pi(q, \omega)$  is only possible if there are some gapless excitations in the system, which is just the case in the QHX state (cf. Eq. (42)). In general, an appearance of gapless excitations with nonzero momentum (low-momenta excitations in our system always have large plasmon gap  $\omega_0$ ) may only be a result of a phase transition, which is associated with spontaneous breakdown of the Galilei invariance. Generally speaking, the Galilei invariance may be spontaneously broken at  $\alpha = \alpha_G < \alpha_{\text{cr}}$ , which would mean the existence of two phase transitions and some intermediate state between the SF and the QHX states.

## VIII. BETWEEN SUPERFLUIDITY AND QUANTUM HEXATIC: SPECULATIONS

We found in previous Sections that at small  $\alpha$  the ground-state  $\Psi_0$  possesses total superfluidity ( $n_s = n$ ) and the excitation spectrum beyond the ground state is gapful (cf. Eq. (40)). At very large  $\alpha \geq \alpha_{\text{cr}} \approx 35$  the ground state is not superfluid at all,  $n_s = 0$ , and the excitation spectrum is very soft (cf. Eq. (42)). How does the transition(s) between these two qualitatively different ground states occur? The most trivial possibility would be that of the direct first-order phase transition at  $\alpha = \alpha_{\text{cr}}$ . This would also mean that the first-order melting/crystallization transition happens in the classical 2D Coulomb gas at the interaction parameter  $\Gamma = 4\alpha \approx 140$ . However, the persistence of total superfluidity up to  $\alpha \approx 35$  looks rather improbable since at so large values of  $\alpha$  the characteristic energy of the

interaction between particles,  $g^2/2\pi$ , exceeds the kinetic energy  $E_{\text{kin}} = \pi\alpha n/m$  by more than 2 orders of magnitude. Note also, that the above-cited value of  $E_{\text{kin}}$  is about 100 times larger than the naive estimate  $n/m$ ; it means that particles are almost localized in rather small regions of space (much smaller than the mean interparticle distance  $n^{-1/2}$ ) and diffuse from these regions on much longer time scale only. Therefore we suspect that at least two phase transitions take place between the totally SF state and the QHX state.

As we discussed above, the QHX quantum ground-state is seen as a 2D Berezinsky crystal with logarithmic interaction as long as only the equal-time density correlations are considered. The nature of melting in such a 2D classical crystal is not quite clear. The old numerical simulations reported weak first-order transition [23], but the recent simulations on the 2D systems with other types of interactions [25] reveal that to discern between 1st and 2nd order melting one probably would need to simulate systems with a huge number of particles, which was not yet done for the system of interest here.

If melting of such a 2D crystal occurs via Halperin-Nelson mechanism, then there exists a usual hexatic state which possesses quasi-long-range orientational order, i.e. power-law decay of the orientational order parameter

$$\langle \exp[6i(\theta(0) - \theta(\mathbf{r}))] \rangle \propto r^{-\eta_6} \quad (44)$$

without any translational order. The corresponding quantum ground-state would be characterized, apart from orientational quasi-LRO, by the isotropic structure factor  $S(p)$  and by the existence of some gapless mode related with a slow rotation of the  $\theta(\mathbf{r})$  angle. The other possibility is that the melting of the 2D crystal with log-interaction occurs via a first-order phase transition, without hexatic phase; even in this case there is a room for a phase transition related to the breakdown of the Galilei invariance, which would be manifested by an appearance of the gapless mode in the excitation spectrum. Indeed, the absence of normal density  $n_n$  in the ground-state of the system with purely potential interactions is the consequence of the Galilei invariance of the Hamiltonian, which is certainly respected in the low- $\alpha$  limit. An appearance of non-zero  $n_n$  is thus related to spontaneous breakdown of the Galilei invariance, which may occur at some value of  $\alpha$  which we denote as  $\alpha_G$ . As it was explained above, we expect  $\alpha_G < \alpha_{\text{cr}}$ , although we still cannot prove this inequality.

To conclude this Section: if there exists any additional ground-state sandwiched between SF and QHX, then the spectrum of excitations beyond such a state contains a gapless mode with isotropic spectrum:  $\omega(p) \propto (p - p_0)^\beta$ . Such a mode may be orthogonal to the density mode and, thus, it may be invisible in the usual structure factor  $S(q, \omega)$ .

## IX. CONCLUSIONS

We have studied the model Hamiltonian of the two-dimensional strongly interacting Bose-liquid, which can be considered as a natural generalization of the Calogero-Sutherland Hamiltonian for the 2D case. This Hamiltonian is shown to possess the non-relativistic supersymmetry. It makes possible to derive a number of exact relations between the equal-time density-density and current-current correlation functions, and to find exactly the low-momentum excitation spectrum of the model, which is given by the one-loop correction to the mean-field result. The ground state is shown to exist in at least two qualitatively different phases as function of the interaction parameter  $\alpha$ : the superfluid state at small  $\alpha$  and the quantum hexatic state at very large  $\alpha$ . The arguments are presented in favor of the existence of a third state, intermediate between the two above-mentioned ones. For the special case of  $\alpha = 1/2$  the considered model is shown to be equivalent to the quantum mechanics of normal complex matrices with a Gaussian weight, exactly in the same way as the Calogero-Sutherland model at the special values of its coupling constant is equivalent to the dynamics of the Wigner-Dyson matrix ensembles.

The model we have studied seems to be very interesting as a natural approximation for the problem of 2D Bose-liquid with logarithmic pair-wise interaction, which is widely considered in relation with the composite-boson approach to the description of the Fractional Quantum Hall state [8,27], as well as in relation with statistics of vortex lines in HTSC superconductors [1,7]. The key physical problem here is the problem of existence of non-superfluid liquid ground-state. We have shown within the considered model the existence of at least one nontrivial ground-state, which is quantum hexatic, i.e. anisotropic non-superfluid liquid. It is still an open question if the destruction of superfluidity may occur without *any* explicit symmetry-breaking in the real space, i.e. if it might be that the only spontaneously broken symmetry is the Galilei invariance. To answer this question it is necessary to study the spectrum of lowest-energy excitations at large wavevectors  $q \sim 2\pi\sqrt{n}$  and at large  $\alpha \in [2, 35]$  and look for the appearance of some gapless mode; note that such a mode, if it exists, may be orthogonal to the density-oscillation mode, and thus may be absent from the dynamic structure factor  $S(q, \omega)$ .

The exact results obtained make it possible to suspect that the considered model can actually be exactly solvable, at least at some special values of  $\alpha$ . There are several directions in which one could try to extend and develop the

results presented here: i) to find the time-dependent correlation functions, at least for  $\alpha = 1/2$ , by the solution of the normal-matrix problem; ii) to generalize the correspondence between our Hamiltonian and the normal matrix problem for other values of  $\alpha$ ; iii) to develop an explicitly supersymmetric second-quantized many-body theory and the corresponding diagram technique for our Hamiltonian; this might be very helpful in order to generalize our result for the spectrum, Eq. (39), for an arbitrary momenta  $q$ . A completely different direction of study of this model is provided by its equivalence to a Fokker-Planck equation of multi-particle stochastic dynamics (cf. Appendix), which makes it possible to use *classical* Monte-Carlo simulations in order to study some properties of the quantum many-body problem in question.

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## APPENDIX: SUSY, LANGEVIN DYNAMICS AND NUMERICAL SIMULATIONS

The relation between supersymmetric quantum mechanics and Fokker-Planck equation of stochastic classical dynamics is well-known now [13,26] (cf. also recent preprint [22]). In this Appendix we will show how to use that equivalence in order to implement fast numerical simulation of the many-body quantum-mechanical problem we are interested in.

Consider a system of  $N$  coupled Langevin equations:

$$\gamma \frac{dr_{j,\alpha}}{dt} = -\frac{\partial W\{\mathbf{r}_i\}}{\partial r_{j,\alpha}} + \xi_{j,\alpha}(t) \quad (45)$$

with a Gaussian distribution of random forces  $\xi(t)$ :

$$\overline{\xi_{j,\alpha}(t)\xi_{k,\beta}(t')} = 2\gamma\mathcal{T}\delta_{jk}\delta_{\alpha\beta}\delta(t-t'). \quad (46)$$

Here  $\mathcal{T}$  is temperature of a thermal bath,  $\gamma$  is a damping coefficient which can be set to 1 by a proper rescaling of time.

The Fokker-Planck equation for the probability density function,

$$\gamma \frac{\partial \mathcal{P}\{\mathbf{r}_i\}}{\partial t} = \sum_{j,\alpha} \frac{\partial}{\partial r_{j,\alpha}} \left( \frac{\partial W}{\partial r_{j,\alpha}} + \mathcal{T} \frac{\partial}{\partial r_{j,\alpha}} \right) \mathcal{P}\{\mathbf{r}_i\}, \quad (47)$$

after the substitution  $\mathcal{P}\{\mathbf{r}_i\} = e^{-W/2\mathcal{T}} \Psi\{\mathbf{r}_i\}$  reduces to an imaginary-time Shrödinger equation

$$\gamma \frac{\partial \Psi}{\partial t} = \sum_{j,\alpha} \left\{ \mathcal{T} \frac{\partial^2 \Psi}{\partial r_{j,\alpha}^2} + \left[ \frac{1}{2} \frac{\partial^2 W}{\partial r_{j,\alpha}^2} - \frac{1}{4\mathcal{T}} \left( \frac{\partial W}{\partial r_{j,\alpha}} \right)^2 \right] \Psi \right\}. \quad (48)$$

The many-particle ground state of Eq. (48) has zero energy, which is a consequence of a supersymmetry [13], and wavefunction  $\Psi_0 = \text{const } e^{-W\{\mathbf{r}\}/2\mathcal{T}}$ .

An equivalence of Langevin, Fokker-Planck and Shrödinger approaches implies that the averaging over many-particle ground state  $\Psi_0$  is equivalent to the averaging over the Langevin dynamics of particles (with respect to random force and initial particles distribution) [22]:

$$\langle \Psi_0 | F\{\mathbf{r}_i\} | \Psi_0 \rangle = \overline{F\{\mathbf{r}_i\}}_L. \quad (49)$$

The relation (49) is very important as it makes possible to compute ground-state averages by numerical simulation of Langevin dynamics with a proper interparticle interaction  $W$  which is technically substantially simpler than direct quantum Monte-Carlo simulations. The problem at hand, given by the wavefunction (5), corresponds to a stochastical classical dynamics of a one-component 2D Coulomb gas governed by the following potential energy:

$$W\{\mathbf{r}_i\} = -4\alpha\mathcal{T} \sum_{j>k} \ln |\mathbf{r}_j - \mathbf{r}_k| + 2\pi\alpha\mathcal{T}n \sum_i r_i^2. \quad (50)$$

We have performed Monte-Carlo computer simulations of the Brownian dynamics (45) with the potential (50) for the systems of  $N = 109$  particles with different values of the parameter  $\alpha$ . The results obtained for the structure factor  $S(p)$  are shown in Fig. 1. Also shown are the curves for the correlation function of transverse currents,  $\Pi_{\perp}(p)$ , calculated from  $S(p)$  according to Eq. (29).

We have also used this simulation in order to check Eq. (27) for different values  $\alpha$ . Note that the general relation (49) can not be applied to compute the mean value of the kinetic energy since the operator of the kinetic energy contains space derivatives that have no immediate meaning in the classical Langevin dynamics. Fortunately, instead of  $E_{\text{kin}}$  one can compute the mean ground-state potential energy  $E_{\text{pot}}$  using the fact that the ground-state energy is zero.  $E_{\text{pot}}$  may be easily obtained by Langevin simulations using Eq. (49) since the operator of the potential energy  $U_{\text{KKLZ}} + E^0 + \frac{(2\pi\alpha n)^2}{2m} \sum_i r_i^2$  contains particles coordinates only.

Numerical simulation have showed that Eq. (27) holds for all values of  $\alpha$  from  $\leq 1$ , that corresponds to the SF state, to 100, corresponding to the CR state, within the precision of computation.

- [1] M. V. Feigel'man, V. B. Geshkenbein, L. B. Ioffe and A. I. Larkin, Phys.Rev.**B48**, 16641 (1993).
- [2] G. Blatter, M. V. Feigel'man, V. B. Geshkenbein, A. I. Larkin and V. M. Vinokur, Rev.Mod.Phys. **66**, 1125 (1994).
- [3] P. W. Anderson, Science **235**, 1196 (1987).
- [4] L. B. Ioffe and A. I. Larkin Phys. Rev. **B 39**, 8988 (1989).
- [5] G. Baskaran "Fermi condensation of bosons in the  $t - J$  model", talk given at the ICTP workshop (1996).
- [6] Y.-H. Li and S. Teitel, Phys.Rev.**B 47**, 359 (1993); *ibid* **B 49**, 4136 (1994); Tao Chen and S. Teitel, preprint cond-mat/9610151.
- [7] H. Nordborg and G. Blatter, preprint cond-mat/9612025
- [8] C. Kane, S. Kivelson, D.-H. Lee and S. C. Zhang, Phys.Rev.**B43**, 3255 (1991).
- [9] B. I. Simons, P. A. Lee and B. L. Altshuler, Nucl. Phys. **B 409**, [FS] 487 (1993).
- [10] N. Taniguchi, B. S. Shastry and B. L. Altshuler, Phys.Rev.Lett. **75**, 3724 (1995).
- [11] B. I. Halperin and D. R. Nelson, Phys.Rev.**B 19**, 2457 (1979).
- [12] E. Witten, Nucl. Phys. **B 188**, 513 (1981).
- [13] M. V. Feigel'man and A. M. Tsvelick, ZhETF **83**, 1430 (1982) [Sov.Phys.JETP **56**, 823 (1982)].
- [14] V. L. Berezinsky ZhETF **59**, 907 (1970) [Sov.Phys.JETP **32**, 493 (1971)].
- [15] D. J. Gross, in "Two-dimensional quantum gravity and random surfaces", v.8, Proc. of the Jerusalem Winter School 1991, Eds. D. J. Gross, T. Piran and S. Weinberg, World Scientific Publ. Co., 1992.
- [16] M. L. Mehta, *Random Matrices*, 2nd Edition (Academic Press 1991).
- [17] D. Gross and I. Klebanov, Nucl.Phys. **B352**, 671 (1991).
- [18] J. Ginibre, J.Math.Phys. **6**, 440 (1965).
- [19] A. Khare and K. Ray, preprint hep-th/9609025.
- [20] G. Oas, Phys.Rev.**E 55**, 205 (1997).
- [21] if it would be the case, one could try to solve the dynamical problem by the generalization of the method employed in [9], i.e. by the mapping onto the appropriate Supersymmetric Sigma Model.
- [22] P. J. Forrester and B. Jancovici, preprint cond-mat/9610085.
- [23] Ph. Choquard and J. Clerouin, Phys.Rev.Lett.**50**, 2086 (1983).
- [24] M. A. Skvortsov and M. V. Feigel'man, in preparation.
- [25] K. Bagchi, H. C. Andersen and W. Swope, Phys.Rev.Lett. **76**, 255 (1996).
- [26] J. Zinn-Justin, *Quantum Field theory and Critical Phenomena*, Clarendon Press, Oxford (1993).
- [27] M. P. A. Fisher and D.-H. Lee, Int.J.Mod.Phys **5**, 2675 (1992).